Combining Separation of Variables & Coulomb Integrals

- Coulomb integrals are often difficult to evaluate. We get stuck unless we have very symmetric distributions of charge and we ask for V or $\vec{E}$ @ a point that doesn't spoil that symmetry.

- For example, given a charge density $\sigma = \sigma_0 \cos\theta$ on a sphere, I can easily evaluate the Coulomb integral to find V @ a point on the z-axis:

$$V(0,0,z) = \begin{cases} \dfrac{\sigma_0 z}{3\varepsilon_0}, & |z| < R \\ \dfrac{\sigma_0 R^3}{3\varepsilon_0 z^2} \dfrac{z}{|z|}, & |z| > R \end{cases}$$

In class we assumed $z>0$. If $z<0$ we get a $-$ sign!
$$\frac{z}{|z|} = \begin{cases} +, & z>0 \\ -, & z<0 \end{cases}$$

- But as we've seen, we can use S.o.V. to find the potential anywhere for this distribution of charge. So why worry about Coulomb integrals?

- We can actually use the result above, obtained from a Coulomb integral, to identify the coefficients in our S.o.V. solution. Here it's overkill, but this sort of "Hybrid" approach is very useful in other cases.

- Let's see how it works. Here, we know from azimuthal symmetry that the potential can be written

$$V(r,\theta) = \begin{cases} \displaystyle\sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \cancelto{0}{B_\ell \frac{1}{r^{\ell+1}}}\right) P_\ell(\cos\theta), & r \le R \\ \displaystyle\sum_{\ell=0}^{\infty} \left(\cancelto{0}{C_\ell r^\ell} + D_\ell \frac{1}{r^{\ell+1}}\right) P_\ell(\cos\theta), & r \ge R \end{cases}$$

- Let's compare these to our expression for V @ a point on the z-axis.

- The z-axis is $\theta = 0$ $(z > 0)$ or $\theta = \pi$ $(z < 0)$ in SPC.
- First we'll consider $|z| < R$. For $z > 0$, $\theta = 0$ and $P_\ell(\cos 0) = P_\ell(1) = 1$ $\forall\, \ell$. We get

$$\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos 0) = \sum_{\ell=0}^{\infty} A_\ell r^\ell = \frac{\sigma_0}{3\varepsilon_0} r \quad z = r \text{ @ } \theta = 0$$

$$\rightarrow A_1 = \frac{\sigma_0}{3\varepsilon_0}, \quad A_\ell = 0 \quad \forall\, \ell \geq 2$$

- If we compared @ a point $z < 0$ & $|z| < R$, we'd have $\theta = \pi$. Then $P_\ell(\cos\pi) = P_\ell(-1) = (-1)^\ell$ $\forall\, \ell$, and $z = -r$:

$$\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\pi) = \sum_{\ell=0}^{\infty} A_\ell r^\ell (-1)^\ell = \frac{\sigma_0}{3\varepsilon_0} \cdot (-r)$$

$$\rightarrow A_1 = \frac{\sigma_0}{3\varepsilon_0}, \quad A_\ell = 0 \quad \forall\, \ell \geq 2$$

- So knowing $V(0,0,z)$ for $|z| < R$ is enough to tell us that our S.o.V. inside the sphere is

$$V(r<R, \theta) = \frac{\sigma_0}{3\varepsilon_0} r\cos\theta$$

- What about outside the sphere? First consider $z > 0$.

$$\sum_{\ell=0}^{\infty} D_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos 0) = \sum_{\ell=0}^{\infty} D_\ell \frac{1}{r^{\ell+1}} = \frac{\sigma_0}{3\varepsilon_0} \frac{R^3}{r^2} \quad z = r \text{ @ } \theta = 0$$

$$\rightarrow D_1 = \frac{\sigma_0 R^3}{3\varepsilon_0}, \quad D_\ell = 0 \quad \forall\, \ell \geq 2$$

- Let's check $\theta = \pi$ $(z<0)$ as well. There, $z = -r$.

$\frac{z}{|z|}$ for $z<0$

$$\sum_{\ell=0}^{\infty} D_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\pi) = \sum_{\ell=0}^{\infty} D_\ell \frac{1}{r^{\ell+1}} \cdot (-1)^\ell = \frac{\sigma_0}{3\varepsilon_0} \cdot \frac{R^3}{(-r)^2} \cdot (-1)$$

$(-r)^2 = r^2$

$$\rightarrow D_1 = \frac{\sigma_0 R^3}{3\varepsilon_0}, \quad D_\ell = 0 \quad \forall\, \ell \geq 2$$

- Again, the results for $\theta = 0$ & $\theta = \pi$ agree, so:

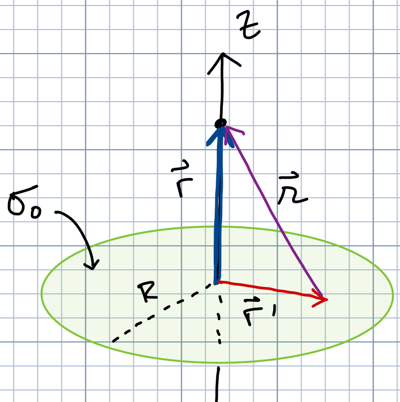
$$V(r>R, \theta) = \frac{\sigma_0}{3\varepsilon_0} \frac{R^3}{r^2} \cos\theta$$

- Of course we arrive @ the same result we got when we solved this as a S.o.V. boundary value problem.

- Can we apply this to something more complicated?

- Our first example of a Coulomb integral was the electric field above or below the center of a uniformly charged disk. In that calculation we made that line through the center of the disk our z-axis. Later we calculated $V(0,0,z)$. Can we use this hybrid approach to find V everywhere? (Yes.)

- First, what was $V(0,0,z)$?



$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi' \int_0^R ds'\, s'\, \sigma_0 \frac{1}{\sqrt{s'^2 + z^2}}$$

$$= \frac{\sigma_0}{2\varepsilon_0}\int_{z^2}^{R^2+z^2} du\, \frac{1}{2}\frac{1}{\sqrt{u}} = \frac{\sigma_0}{2\varepsilon_0}\sqrt{u}\,\Big|_{z^2}^{R^2+z^2}$$

$$\hookrightarrow V(0,0,z) = \frac{\sigma_0}{2\varepsilon_0}\left(\sqrt{R^2+z^2} - \sqrt{z^2}\right)$$

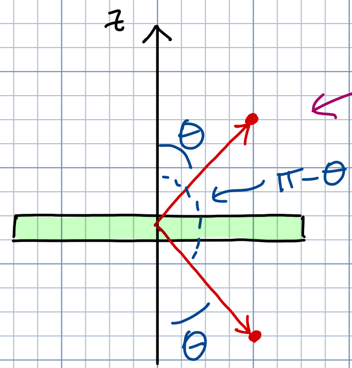
- When evaluating this, we'd be careful to write $\sqrt{z^2}$ as z for $z>0$ & $-z$ for $z<0$.

- Of course, $\sqrt{z^2}$ is just r here — the distance from the origin. So we could also write $V(0,0,z)$ as

$$V(r,\theta=0) = V(r,\theta=\pi) = \frac{\sigma_0}{2\varepsilon_0}\left(\sqrt{R^2+r^2} - r\right)$$

- Now what about V <u>off-axis</u>? First, we expect it to be symmetric across the x-y plane.

- Consider two points above & below the disk:



Both pts are the same distance from each bit of charge on the disk, so $V(r,\theta) = V(r,\pi-\theta)$.

- Since we know that $V(r,\pi-\theta) = V(r,\theta)$, we will just work out $V(r,\theta)$ for $0 \le \theta < \pi/2$.

- The disk is an azimuthally symmetric distribution of charge, and we know what the potential looks like in that case – powers of r & $1/r$ along w/ Legendre polynomials. But where do I see that in the $V(0,0,z)$ we found from the Coulomb integral?

$$V(r,\theta=0) = \frac{\sigma_0}{2\varepsilon_0} \times \left(\sqrt{R^2+r^2} - r\right)$$

Expand in powers of r/R for $r<R$, and powers of R/r for $r>R$

TAYLOR: $$\sqrt{1+\epsilon^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!}\,\epsilon^{2n}$$

$$V(r<R,\theta=0) = \frac{\sigma_0}{2\varepsilon_0} \times \left(R\sqrt{1+\frac{r^2}{R^2}} - r\right)$$

$$= \frac{\sigma_0}{2\varepsilon_0} \times \left(-r + R\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \cdot \frac{r^{2n}}{R^{2n}}\right)$$

$$V(r>R,\theta=0) = \frac{\sigma_0}{2\varepsilon_0} \times \left(r\sqrt{1+\frac{R^2}{r^2}} - r\right)$$

$$= \frac{\sigma_0}{2\varepsilon_0} \times \left(-r + r\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \cdot \frac{R^{2n}}{r^{2n}}\right)$$

The $n=0$ term is '1', so $r\cdot 1$ cancels the $-r$ term

$$= \frac{\sigma_0}{2\varepsilon_0} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \cdot \frac{R^{2n}}{r^{2n-1}}$$

Re-index $n \to n+1$

$$= \frac{\sigma_0}{2\varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^{n}(2n-1)!!}{2^{n+1} \cdot (n+1)!} \cdot \frac{R^{2n+2}}{r^{2n+1}}$$

- So on the z-axis we can write V as an expansion in powers of r/R for $r < R$, and in powers of R/r for $r > R$. We just need to compare these to the S.o.V. solution on the z-axis.

$$V(r<R,\theta=0) = \sum_{\ell=0}^{\infty} A_\ell r^\ell \overbrace{P_\ell(\cos 0)}^{=1\,\forall \ell} = \frac{\sigma_0}{2\varepsilon_0} \times \left(-r + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \frac{r^{2n}}{R^{2n-1}}\right)$$

$$\Rightarrow A_1 = -\frac{\sigma_0}{2\varepsilon_0} \qquad A_{2n+1} = 0 \quad \forall\ n \geq 1$$

$$A_{2n} = \frac{\sigma_0}{2\varepsilon_0} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \frac{1}{R^{2n-1}}$$

$$V(r>R,\theta=0) = \sum_{\ell=0}^{\infty} D_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos 0) = \frac{\sigma_0}{2\varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^{n+1} \cdot (n+1)!} \cdot \frac{R^{2n+2}}{r^{2n+1}}$$

$$\Rightarrow D_{2n} = \frac{\sigma_0}{2\varepsilon_0} \frac{(-1)^n (2n-1)!!}{2^{n+1} \cdot (n+1)!} \cdot R^{2n+2} \qquad D_{2n+1} = 0 \quad \forall\ n \geq 0$$

- Now, we know $V(r, \pi-\theta) = V(r,\theta)$ b/c the potential should look the same on either side of the disk. That's good, b/c <u>most</u> of the terms we found above (A_{2n} & D_{2n}) show up w/ even Legendre polynomials. So most of the terms in our S.o.V. solution give the same value for θ & $\pi - \theta$, since $P_{2n}(\cos(\pi-\theta)) = P_{2n}(-\cos\theta) = (-1)^{2n} P_{2n}(\cos\theta) = P_{2n}(\cos\theta)$.

- But there's one exception! It's the $\ell = 1$ term for $r < R$. That term flips sign for $\pi - \theta$:

$$-\frac{\sigma_0}{2\varepsilon_0} r \cdot P_1(\cos(\pi-\theta)) = -\frac{\sigma_0}{2\varepsilon_0} r \cdot \cos(\pi-\theta) = +\frac{\sigma_0}{2\varepsilon_0} r\cos\theta$$

So when we write our S.o.V. solution for $\pi/2 < \theta \leq \pi$, we have to flip the sign of the $\ell = 1$ term for $r < R$. (We will see why in just a moment.)

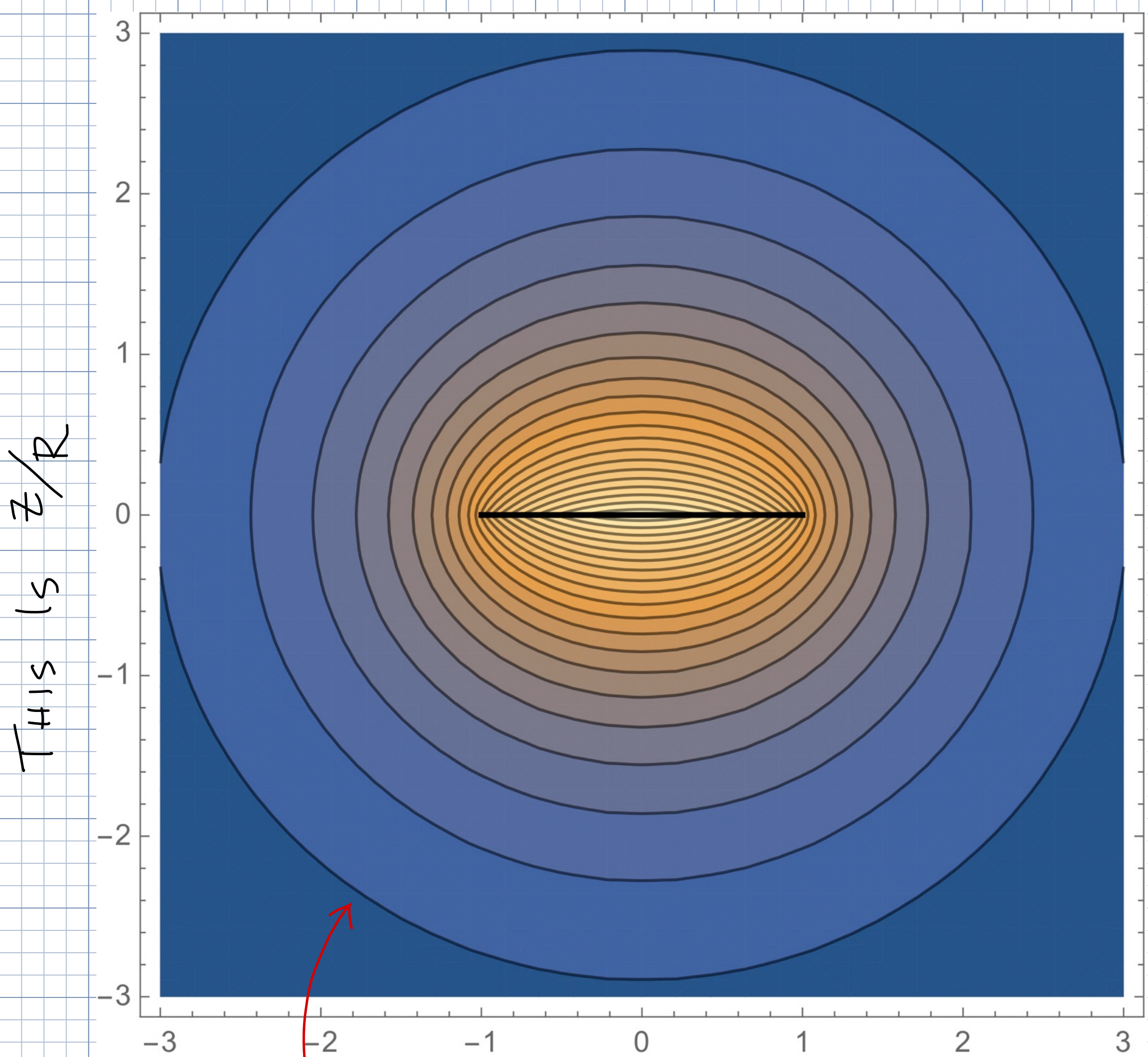
- The potential due to the disk @ any point $\vec{r}$ is:

$$V(r<R, 0\leq\theta<\pi/2) = -\frac{\sigma_0}{2\varepsilon_0} r P_1(\cos\theta) + \sum_{n=0}^{\infty} \frac{\sigma_0}{2\varepsilon_0} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \frac{r^{2n}}{R^{2n-1}} P_{2n}(\cos\theta)$$

$$V(r<R, \tfrac{\pi}{2}<\theta\leq\pi) = +\frac{\sigma_0}{2\varepsilon_0} r P_1(\cos\theta) + \sum_{n=0}^{\infty} \frac{\sigma_0}{2\varepsilon_0} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \frac{r^{2n}}{R^{2n-1}} \cdot P_{2n}(\cos\theta)$$

$$V(r>R, \theta) = \frac{\sigma_0}{2\varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^{n+1} \cdot (n+1)!} \cdot \frac{R^{2n+2}}{r^{2n+1}} P_{2n}(\cos\theta)$$

- That's it! There are a few subtleties, but first lets plot it.



Curves are surfaces of constant V.

- Now, V is always continuous, so we should be sure that our results for $r \leq R$ & $r \geq R$ agree @ $r = R$. And when we do this, we find something funny!

$$-\frac{\sigma_0}{2\varepsilon_0} R P_1(\cos\theta) + \frac{\sigma_0}{2\varepsilon_0}\sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} R \cdot P_{2n}(\cos\theta) \stackrel{?}{=} \ldots$$

$$\ldots \stackrel{?}{=} \frac{\sigma_0}{2\varepsilon_0}\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n-1)!!}{2^{n+1} \cdot (n+1)!} R \cdot P_{2n}(\cos\theta)$$

Canceling all the common factors, we have

$$-P_1(\cos\theta) + \sum_{n=0}^{\infty} \frac{(-1)^{n-1} \cdot (2n-3)!!}{2^n \cdot n!} \cdot \frac{(4n+1)}{2(n+1)} \cdot P_{2n}(\cos\theta) \stackrel{?}{=} 0$$

- But how can this be? How can a bunch of Legendre polynomials add up to zero? Aren't they orthogonal?

- Yes, the $P_\ell(\cos\theta)$ are orthogonal for $0 \leq \theta \leq \pi$, but our expression for $V(r \leq R, \theta)$ that we used above only applies for $0 \leq \theta < \pi/2$! It's not obvious based on what we learned in Math Methods, but that sum above <u>does</u> add up to $P_1(\cos\theta)$ if we only consider $0 \leq \theta < \pi/2$.

- Likewise, we find that our $r \leq R$ and $r \geq R$ expressions for V also agree @ $r = R$ for $\pi/2 < \theta \leq \pi$.

- How else can we check our result? The disk has a surface charge on it, so $\vec{E}$ should be discontinuous @ $\theta = \pi/2$ & $r \leq R$. Specifically:

$$\vec{E}_{above}\Big|_{Disk} - \vec{E}_{below}\Big|_{Disk} = \frac{\sigma_0}{\varepsilon_0}\hat{z}$$

- Now, we have V as a function of r & θ, and we write it the same way for $\theta < \pi/2$ & $\theta > \pi/2$ when $r \leq R$ <u>except</u>

for the $\ell = 1$ term:

$$V(r \le R,\, 0 \le \theta < \pi/2) = -\frac{\sigma_0}{2\varepsilon_0} \overbrace{r\cos\theta}^{z} + (\text{MORE TERMS})$$

$$V(r \ge R,\, \pi/2 < \theta \le \pi) = +\frac{\sigma_0}{2\varepsilon_0} \underbrace{r\cos\theta}_{z} + (\text{SAME TERMS})$$

$$\hookrightarrow \vec{E}(r < R,\, 0 \le \theta < \pi/2) = \frac{\sigma_0}{2\varepsilon_0}\hat{z} - \vec{\nabla}(\text{MORE TERMS})$$

$$\vec{E}(r < R,\, \pi/2 < \theta \le \pi) = -\frac{\sigma_0}{2\varepsilon_0}\hat{z} - \vec{\nabla}(\text{SAME TERMS})$$

- So when we compare these @ $z = 0$ for $r \le R$ we get

$$\vec{E}_{above}(r \le R, \pi/2) - \vec{E}_{below}(r \le R, \pi/2) = \frac{\sigma_0}{2\varepsilon_0}\hat{z} - \left(-\frac{\sigma_0}{2\varepsilon_0}\hat{z}\right) \overset{\checkmark}{=} \frac{\sigma_0}{\varepsilon_0}\hat{z}$$

- So, by comparing our Coulomb integral result for the potential above & below the center of a charged disk w/ the general S.o.V. solution for an azimuthally symmetric problem, we were able to find $V(r,\theta)$ <u>everywhere</u>, not just along the z-axis ($\theta = 0$ or $\theta = \pi$).

- Confirming that the solution we found is continuous @ $r = R$ requires some properties of Legendre polynomials that go beyond what we covered in Math Methods, but everything works as expected.

- Here's the final result again:

$$V(r \le R,\, 0 \le \theta < \pi/2) = -\frac{\sigma_0}{2\varepsilon_0} r\cos\theta + \sum_{n=0}^{\infty} \frac{\sigma_0}{2\varepsilon_0} \times \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \frac{r^{2n}}{R^{2n-1}} P_{2n}(\cos\theta)$$

$$V(r \le R,\, \pi/2 < \theta \le \pi/2) = +\frac{\sigma_0}{2\varepsilon_0} r\cos\theta + \sum_{n=0}^{\infty} \frac{\sigma_0}{2\varepsilon_0} \frac{(-1)^{n-1}(2n-3)!!}{2^n \cdot n!} \cdot \frac{r^{2n}}{R^{2n-1}} \cdot P_{2n}(\cos\theta)$$

$$V(r > R, \theta) = \frac{\sigma_0}{2\varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^{n+1} \cdot (n+1)!} \cdot \frac{R^{2n+2}}{r^{2n+1}} P_{2n}(\cos\theta)$$